Taylor Series
Single Variable and Multi-Variable

- Single variable Taylor series:
  Let f be an infinitely differentiable function in some open interval around $x = a$.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- Linear approximation in one variable:
  Take the constant and linear terms from the Taylor series. In an open interval around $x = a$,

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{linear approximation}$$

- Quadratic approximation in one variable:
  Take the constant, linear, and quadratic terms from the Taylor series. In an open interval around $x = a$,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \quad \text{quadradic approximation}$$

- Multi variable Taylor series:
  Let f be an infinitely differentiable function in some open neighborhood around $(x, y) = (a, b)$.

$$\begin{aligned} f(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2\right] + \cdots \end{aligned}$$

- A more compact form:
  Let $\mathbf{x} = \langle x, y \rangle$ and let $\mathbf{a} = \langle a, b \rangle$. With this new vector notation, the Taylor series can be written as

$$f(\mathbf{x}) = f(\mathbf{a}) + [(\mathbf{x} - \mathbf{a}) \cdot \boldsymbol{\nabla} f(\mathbf{a})] + [(\mathbf{x} - \mathbf{a}) \cdot (H(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{a}))] + \cdots$$

  where H is the matrix of second derivatives, called the **Hessian matrix**

$$H(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

- Linear approximation in multiple variables:
  Take the constant and linear terms from the Taylor series. In a neighborhood of $(x, y) = (a, b)$,

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- Quadratic approximation in multiple variables:
  Take the constant, linear, and quadratic terms from the Taylor series. In a neighborhood of $(x, y) = (a, b)$,

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2\right] \\ &= f(\mathbf{a}) + [(\mathbf{x} - \mathbf{a}) \cdot \boldsymbol{\nabla} f(\mathbf{a})] + [(\mathbf{x} - \mathbf{a}) \cdot (H(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{a}))] \end{aligned}$$